

# Purely pure de Sitter supergravity

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**Abstract:** This work revisits  $D = 4, N = 1$  (anti)-de Sitter supergravity within a deliberately unconventional framework aiming to show that  $D = 4, N = 1$  de Sitter supergravity can be derived only with the graviton and the gravitino fields, and therefore without a Goldstino field, a scalar sector, or spontaneous supersymmetry breaking. The unconventional features of this work consist in using the Majorana conjugate to construct bilinears of Weyl spinors and introducing the hyperbolic unit in the fermionic terms related to a positive cosmological constant.

“One should not desist from pursuing to the end the path of the relativistic field theory.”  
A. Einstein

“A great deal of my work is just playing with equations and seeing what they give.”  
P. Dirac

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The last version of this work can be found at <https://ds-supergravity.org/>.

## 1 Introduction

Supergravity is a wonderful achievement at the crossroads between the works of A. Einstein and P. Dirac [1–4]. Noteworthily, almost since its discovery fifty years ago,  $D = 4, N = 1$  supergravity is known to be anti-de Sitter with a negative cosmological constant when only the graviton bosonic field  $e_\mu^m$  and the gravitino fermionic field  $\psi_\mu$  are considered [5]. It is only recently that  $D = 4, N = 1$  de Sitter supergravity with a positive cosmological constant has been derived by adding a nilpotent Goldstino fermionic field  $\chi$  to obtain the local supersymmetry and then by eliminating it through spontaneous supersymmetry breaking [6].

Given the no-go theorems on the subject [7,8], it is clear that hoping to derive  $D = 4, N = 1$  de Sitter supergravity in a different way than [6] requires an unconventional approach.

For motivation to derive  $D = 4, N = 1$  de Sitter supergravity, the reader is referred to the Introduction of [6].

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## 2 A reminder on spinors, Lorentz transformations, local Lorentz derivative, and spinor bilinears

### 2.1 Spinors

This section follows Sec. 3.2 of [4]. Given any spinor  $\chi \equiv (\chi)_\alpha$  whose components are four anti-commuting Grassmann variables, its Dirac conjugate  $\bar{\chi}^{(\text{Dirac})} \equiv (\chi)^\alpha$  and its charge conjugate  $\chi^C \equiv (\chi^C)_\alpha$  are defined by<sup>1</sup>

$$\bar{\chi}^{(\text{Dirac})} \equiv i\chi^\dagger \gamma^0, \quad (1)$$

$$\chi^C \equiv i\gamma^0 C^\dagger \chi^*, \quad (2)$$

where  $C$  is the charge conjugation matrix having the following properties<sup>2</sup>

$$\begin{aligned} C^\dagger &= C^{-1}, \quad C^T = -C \implies CC^* = C^*C = -\mathbb{1}, \\ (C\gamma_*)^T &= -C\gamma_*, \\ (C\gamma^m)^T &= C\gamma^m, \\ (C\gamma^{mn})^T &= C\gamma^{mn}, \\ (C\gamma^{mnr})^T &= -C\gamma^{mnr}, \\ (C\gamma^{mnr s})^T &= -C\gamma^{mnr s}. \end{aligned} \quad (3)$$

From the definitions (1),(2) and the properties (3), one can verify that

$$\overline{\chi^C}^{(\text{Dirac})} \equiv i(\chi^C)^\dagger \gamma^0 = \chi^T C, \quad (4)$$

which can be used to define the Majorana conjugate of any spinor  $\chi$  by

$$\bar{\chi}^{(\text{Majorana})} \equiv \chi^T C. \quad (5)$$

In this work, the Majorana conjugate (5) will be used and not the Dirac conjugate (1) for constructing bilinears of Weyl spinors as explained below.

### 2.2 Lorentz transformations

The Lorentz transformations of a vector  $V^m$  and a spinor  $\chi$  are defined by<sup>3</sup>

$$\delta_L V^m = -\omega^m{}_k V^k \iff \delta_L V_m = -\omega_m{}^k V_k, \quad (6)$$

$$\delta_L \chi = -\frac{1}{4}\omega_{ab}\gamma^{ab}\chi \iff \delta_L \bar{\chi} = \frac{1}{4}\omega_{ab}\bar{\chi}\gamma^{ab}, \quad (7)$$

where  $\omega_{ab}$  is an infinitesimal Lorentz transformation that is antisymmetric  $\omega_{ab} = -\omega_{ba}$

<sup>1</sup>The conventions of this work are those of [4] except for the definition of  $\gamma_*$ : the metric signature is  $(-+++)$ ; the four  $\gamma$ -matrices are defined by  $\gamma^m\gamma^n + \gamma^n\gamma^m = 2\eta^{mn}\mathbb{1}$  where  $\mathbb{1}$  is the unit matrix  $\implies (\gamma^0)^2 = -\mathbb{1}$  and  $(\gamma^k)^2 = \mathbb{1}$  with  $k = 1, 2, 3$ ;  $(\gamma^m)^\dagger = \gamma^0\gamma^m\gamma^0 \implies (\gamma^0)^\dagger = -\gamma^0$ ,  $(\gamma^k)^\dagger = \gamma^k$ ;  $\gamma_m \equiv \eta_{mn}\gamma^n \implies \gamma_0 = -\gamma^0$  and  $\gamma_k = \gamma^k$ ;  $\gamma_* \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \implies (\gamma_*)^2 = \mathbb{1}$ ,  $(\gamma_*)^\dagger = \gamma_*$  and  $\gamma_*\gamma^m = -\gamma^m\gamma_*$ ;  $\gamma^{mn} \equiv \gamma^{[m}\gamma^{n]}$  antisymmetric with strength one, and so on for  $\gamma^{mnr}$  and  $\gamma^{mnr s}$ ;  $\epsilon^{\mu\nu\rho\sigma} \equiv e^\mu{}_{e_m} e^\nu{}_{e_n} e^\rho{}_{e_r} e^\sigma{}_{e_s} \implies \epsilon_{\mu\nu\rho\sigma} = e^{-1}\epsilon_{mnrs} e_\mu^m e_\nu^n e_\rho^r e_\sigma^s$  with  $\epsilon_{0123} = +1 = -\epsilon^{0123}$ .

<sup>2</sup>The Weyl representation of the  $\gamma$ -matrices is given in Appendix A.

<sup>3</sup>The expression for  $\delta_L \bar{\chi}$  follows directly from (5) and the property  $(\gamma^{ab})^T C = -C\gamma^{ab}$  coming from (3):  $\overline{(\delta_L \chi)} \equiv (\delta_L \chi)^T C = \delta_L(\chi^T)C = \delta_L(\chi^T C) = \delta_L \bar{\chi}$  with  $(\delta_L \chi)^T C = (-\frac{1}{4}\omega_{ab}\gamma^{ab}\chi)^T C = -\frac{1}{4}\omega_{ab}\chi^T(\gamma^{ab})^T C = \frac{1}{4}\omega_{ab}\chi^T C\gamma^{ab} = \frac{1}{4}\omega_{ab}\bar{\chi}\gamma^{ab}$ .

### 2.3 Majorana spinors

A Majorana spinor is defined by the condition

$$\chi^C = \chi. \quad (8)$$

From (4),(5) one can see that the Majorana conjugate of a Majorana spinor is equal to its Dirac conjugate

$$\chi^C = \chi \implies \bar{\chi}^{(\text{Majorana})} \equiv \bar{\chi}^{(\text{Dirac})}. \quad (9)$$

### 2.4 Weyl spinors

Any spinor  $\chi$  can be split into its left and right Weyl projections that are also spinors<sup>4</sup>

$$P_L\chi \equiv \frac{1}{2}(\mathbb{1} - \gamma_*)\chi, \quad (10)$$

$$P_R\chi \equiv \frac{1}{2}(\mathbb{1} + \gamma_*)\chi, \quad (11)$$

with obviously  $\chi = P_L\chi + P_R\chi$ .

In  $D = 4$  it is known that a spinor  $\chi$  cannot be a Majorana and a Weyl spinor at the same time<sup>5</sup>:  $\chi = \chi^C \implies \chi \neq P_L\chi$  and  $\chi \neq P_R\chi$ .

### 2.5 Local Lorentz derivative

Supergravity uses the local Lorentz derivative which acts on Lorentz local frame (Latin) indices and spinor indices but not on general coordinate (Greek) indices. The local Lorentz derivative of a vector  $V^m$  and a spinor  $\chi$  are defined by<sup>6</sup>

$$D_\mu V^m = \partial_\mu V^m + \omega_\mu{}^m{}_k V^k \implies D_\mu V_m \equiv \partial_\mu V_m + \omega_{\mu m}{}^k V_k, \quad (12)$$

$$D_\mu \chi = \partial_\mu \chi + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\chi \implies D_\mu \bar{\chi} = \partial_\mu \bar{\chi} - \frac{1}{4}\omega_{\mu ab}\bar{\chi}\gamma^{ab}, \quad (13)$$

where  $\omega_{\mu mn}$  is the spin connection that is antisymmetric  $\omega_{\mu mn} = -\omega_{\mu nm}$ .

### 2.6 Spinor bilinears

The  $\gamma$ -matrices and  $C$  are numerical matrices on which the Lorentz transformation and the Lorentz derivative do not act<sup>7</sup>:

$$\delta_L \gamma^m = 0, \quad D_\mu \gamma^m = 0, \quad (14)$$

$$\delta_L \gamma_* = 0, \quad D_\mu \gamma_* = 0, \quad (15)$$

$$\delta_L C = 0, \quad D_\mu C = 0. \quad (16)$$

<sup>4</sup>Needless to say that a Weyl projection satisfies (7) thanks to the property  $\gamma_*\gamma^{ab} = \gamma^{ab}\gamma_*$ .

<sup>5</sup>This is due to the fact that  $(P_L\chi)^C = P_R\chi^C$  and  $(P_R\chi)^C = P_L\chi^C$  in  $D = 4$  as shown in Appendix C.1.

<sup>6</sup>The expression for  $D_\mu \bar{\chi}$  follows directly from (5) and the properties (3): it is easy to verify that  $\overline{(D_\mu \chi)} \equiv (D_\mu \chi)^T C = D_\mu(\chi^T C) = D_\mu \bar{\chi}$ .

<sup>7</sup>See Sec. 8.3 of [4]. Note that this is not always clearly stated in the literature.

The Lorentz transformation (7) together with the left part of (14),(16) allow to create spinor bilinears that transform as tensorial entities. For instance,  $\bar{\chi}\xi$  transform as a scalar:  $\delta_L(\bar{\chi}\xi) = \delta_L\bar{\chi}\xi + \bar{\chi}\delta_L\xi = \frac{1}{4}\omega_{ab}\bar{\chi}\gamma^{ab}\xi - \frac{1}{4}\omega_{ab}\bar{\chi}\gamma^{ab}\xi = 0$ .

Using the relation  $\gamma^{ab}\gamma^m - \gamma^m\gamma^{ab} = 2\eta^{bm}\gamma^a - 2\eta^{am}\gamma^b$  one can see that  $\bar{\chi}\gamma^m\xi$  transform as a vector:  $\delta_L(\bar{\chi}\gamma^m\xi) = \delta_L\bar{\chi}\gamma^m\xi + \bar{\chi}\gamma^m\delta_L\xi = \frac{1}{4}\omega_{ab}\bar{\chi}\gamma^{ab}\gamma^m\xi - \frac{1}{4}\omega_{ab}\bar{\chi}\gamma^m\gamma^{ab}\xi = \frac{1}{4}\omega_{ab}\bar{\chi}(\gamma^{ab}\gamma^m - \gamma^m\gamma^{ab})\xi = \frac{1}{2}\omega_{ab}\bar{\chi}(\eta^{bm}\gamma^a - \eta^{am}\gamma^b)\xi = \frac{1}{2}\omega_a{}^m\bar{\chi}\gamma^a\xi - \frac{1}{2}\omega_b{}^m\bar{\chi}\gamma^b\xi = -\omega^m{}_k(\bar{\chi}\gamma^k\xi)$ .

Similarly, one can verify that the spinor bilinears  $\bar{\chi}\gamma^{mn}\chi$ ,  $\bar{\chi}\gamma^{mnr}\chi$  and  $\bar{\chi}\gamma^{mnr}s\chi$  transform as tensors.

### 3 Spinor-ansatz bilinears

Starting from any spinor  $\psi$ , let's construct the spinor-ansatz  $\Psi^i$  and its Majorana conjugate  $\bar{\Psi}^i$

$$\begin{aligned}\Psi^1 &\equiv \psi, & \bar{\Psi}^1 &\equiv (\Psi^1)^T C = \bar{\psi}, \\ \Psi^2 &\equiv (\Psi^1)^C = \psi^C, & \bar{\Psi}^2 &\equiv (\Psi^2)^T C = \overline{\psi^C}.\end{aligned}\tag{17}$$

From two spinor-ansatzes  $\varepsilon^i$ ,  $\Psi^i$  let's also define the following spinor-ansatz bilinear which is real by construction<sup>8</sup>

$$M^{ij}(\bar{\varepsilon}^i N \Psi^j), \tag{18}$$

when  $N$  is any matrix obtained from products of  $\gamma^m$  and  $i\gamma_*$  and when  $M$  is a 2x2 numerical matrix whose components have the following properties

$$\begin{aligned}M_{22} &= (M_{11})^*, \\ M_{21} &= (M_{12})^*.\end{aligned}\tag{19}$$

### 4 Weyl spinor-ansatz bilinears

Starting from a left-handed Weyl spinor  $\psi \equiv P_L\psi$ , the spinor-ansatz (17) becomes<sup>9</sup>

$$\begin{aligned}\Psi^1 &\equiv P_L\psi, & \bar{\Psi}^1 &\equiv (\Psi^1)^T C = P_L\bar{\psi}, \\ \Psi^2 &\equiv (\Psi^1)^C = P_R\psi^C, & \bar{\Psi}^2 &\equiv (\Psi^2)^T C = P_R\overline{\psi^C}.\end{aligned}\tag{20}$$

*In the following sections, we will consider only Weyl spinor-ansatz bilinears as defined by (18), (19) and (20).*

### 5 The unconventional derivation framework

The unconventional framework used to revisit  $D = 4, N = 1$  (anti)-de Sitter supergravity is based on the gravitino  $\psi_\mu$  expressed as a left handed Weyl spinor  $\psi_\mu = P_L\psi_\mu$ . Three numerical 2x2 matrices  $\mathbf{S}_1, \mathbf{S}_2, \mathbf{M}_1$  and four conventional real constants  $k_1, k_2, k_3, k_4$  are used and will be specified in the following sections. In this section it is only supposed that the two 2x2 matrices  $\mathbf{S}$  are symmetric:  $\mathbf{S} = \mathbf{S}^T$ .

<sup>8</sup>This is proven in Appendix B.

<sup>9</sup>The consistency of this construction is proven in Appendix C.1.

The action considered is the sum of the Einstein-Hilbert term, the cosmological constant term, the Rarita-Schwinger term, and one gauged term<sup>10</sup>:

$$S = S_{\text{EH}} + S_{\Lambda} + S_{\text{RS}} + S_{\text{g}\Lambda}, \quad (21)$$

where<sup>11</sup>

$$S_{\text{EH}} = k_1 \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}{}^{mn}, \quad (22)$$

$$S_{\Lambda} = \pm 2k_1 \int dx^4 e \Lambda, \quad (23)$$

$$S_{\text{RS}} = -k_2 \int dx^4 e \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \quad (24)$$

$$S_{\text{g}\Lambda} = +2k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu} \Psi_\nu^j), \quad (25)$$

with

$$R_{\mu\nu}{}^{mn} \equiv \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^m{}_r \omega_\nu{}^{rn} - \omega_\nu{}^m{}_r \omega_\mu{}^{rn}, \quad (26)$$

$$D_\mu \Psi_\nu^i \equiv \partial_\mu \Psi_\nu^i + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \Psi_\nu^i \implies D_\mu \bar{\Psi}_\nu^i = \partial_\mu \bar{\Psi}_\nu^i - \frac{1}{4} \omega_{\mu mn} \bar{\Psi}_\nu^i \gamma^{mn}. \quad (27)$$

With the conventions given in footnote 1, the plus (resp. minus) sign of the cosmological constant term (23) corresponds to  $D = 4, N = 1$  anti-de Sitter (resp. de Sitter) supergravity.

Using the so-called 1.5 order formalism, the goal is to show that the action (21)-(27) is invariant  $\delta S = 0$  under the following local supersymmetry transformations<sup>12</sup> involving the Weyl spinor-ansatz  $\epsilon^i \equiv (\epsilon^i)_\alpha$  as supersymmetry parameter<sup>13</sup>:

$$\delta e_\mu^m = k_3 \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^m \Psi_\mu^j) \implies \delta e_m^\mu = -k_3 \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^\mu \Psi_m^j) \implies \delta e = k_3 e \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^\rho \Psi_\rho^j), \quad (28)$$

$$\delta \Psi_\mu^i = D_\mu \epsilon^i + k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \gamma_\mu \epsilon^h \implies \delta \bar{\Psi}_\mu^i = D_\mu \bar{\epsilon}^i - k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \bar{\epsilon}^h \gamma_\mu. \quad (29)$$

As stated in [2] the 1.5 order formalism is nothing else than the Palatini trick of general relativity extended to supergravity. The spin connection  $\omega_\mu{}^{mn}$  is treated as an independent field and one imposes that the variation of the action (21)-(25) with respect to it vanishes:

$$\delta_\omega S = \delta_\omega S_{\text{EH}} + \delta_\omega S_{\text{RS}} = 0, \quad (30)$$

where<sup>14</sup>

$$\delta_\omega S_{\text{EH}} = k_1 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} \delta \omega_\mu{}^{mn} e_\nu^r D_\rho e_\sigma^s, \quad (31)$$

$$\delta_\omega S_{\text{RS}} = -\frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} \delta \omega_\mu{}^{mn} e_\nu^r \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma^s \Psi_\sigma^j), \quad (32)$$

<sup>10</sup>Note that the matrix  $\mathbf{S}_1$  must be block off-diagonal and the matrix  $\mathbf{S}_2$  block diagonal since we are working with Weyl spinors and using the Majorana conjugate (5).

<sup>11</sup>As usual in supergravity:  $e = \det e_\mu^m$ ;  $\epsilon_{\mu\nu\rho\sigma} \equiv e^{-1} \epsilon_{mnr s} e_\mu^m e_\nu^n e_\rho^r e_\sigma^s$ ,  $\epsilon^{\mu\nu\rho\sigma} \equiv e \epsilon^{mnr s} e_\mu^m e_\nu^n e_\rho^r e_\sigma^s$  with  $\epsilon_{0123} = +1 = -\epsilon^{0123}$  and  $\epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} e_\mu^m e_\nu^n e_\rho^r e_\sigma^s = -2e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu)$ ;  $\gamma^{\mu\nu\rho} \equiv \gamma^{mnr} e_\mu^m e_\nu^n e_\rho^r = -e^{-1} \epsilon^{\mu\nu\rho\sigma} i \gamma_* \gamma_\sigma$  with  $\gamma_\sigma \equiv e_\sigma^m \gamma_m$ .

<sup>12</sup>Note that the matrix  $\mathbf{M}_1$  must be block off-diagonal since we are working with Weyl spinors.

<sup>13</sup>The consistency of the local supersymmetry transformation (29) is proven in Appendix C.2.

<sup>14</sup>See Appendix D for the detailed calculations.

which leads to the equation

$$D_{[\rho}e_{\sigma]}^s = \frac{k_2}{4k_1} \mathbf{S}_1^{ij} \lambda (\bar{\Psi}_\rho^i \gamma^s \Psi_\sigma^j). \quad (33)$$

This equation for  $\omega_\mu^{mn}$  can be solved to find

$$\omega_\mu^{mn} = \omega_\mu^{mn}(e) + K_\mu^{mn}(\Psi), \quad (34)$$

where  $\omega_\mu^{mn}(e)$  is the torsionless spin connection

$$\omega_\mu^{mn}(e) = \frac{1}{2} e^{m\rho} (\partial_\mu e_\rho^n - \partial_\rho e_\mu^n) - \frac{1}{2} e^{n\rho} (\partial_\mu e_\rho^m - \partial_\rho e_\mu^m) - \frac{1}{2} e^{m\rho} e^{n\sigma} (\partial_\rho e_\sigma^r - \partial_\sigma e_\rho^r) e_{r\mu}, \quad (35)$$

and  $K_\mu^{mn}(\Psi)$  is the so-called contortion tensor given by

$$K_\mu^{mn}(\Psi) = \frac{k_2}{4k_1} \mathbf{S}_1^{ij} \lambda (\bar{\Psi}_\mu^i \gamma^m \Psi^{jn} - \bar{\Psi}_\mu^i \gamma^n \Psi^{jm} + \bar{\Psi}^{im} \gamma_\mu \Psi^{jn}). \quad (36)$$

Following the spirit of the 1.5 order formalism, the result (33) shall be taken into account in the next variations. It is proven in Appendix D that the next variations give

$$\begin{aligned} \delta_e S_{\text{EH}} &= -\frac{k_1 k_3}{2} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^s \Psi_\sigma^j)}^a. \\ \delta_e S_\Lambda &= \pm 2k_1 k_3 \Lambda \int \overbrace{dx^4 e \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j)}^b. \\ \delta_e S_{\text{RS}} &= -ik_2 k_3 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l). \\ \delta_e S_{g\Lambda} &= -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l). \\ \delta_\Psi S_{\text{RS}} &= +\frac{k_2}{4} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^s \Psi_\sigma^j)}^a + 4k_2 k_4 \sqrt{\Lambda} \int \overbrace{dx^4 e \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j)}^c \\ &\quad - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) \\ &\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l). \\ \delta_\Psi S_{g\Lambda} &= -4k_2 k_4 \sqrt{\Lambda} \int \overbrace{dx^4 e \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j)}^c - 12k_2 (k_4)^2 \Lambda \int \overbrace{dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j)}^b \\ &\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\ &\quad + \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l). \end{aligned} \quad (37)$$

In order to partly fulfill the supersymmetric condition  $\delta S = 0$ , the terms (a),(b),(c) of (37) give<sup>15</sup>

$$k_3 = \frac{k_2}{2k_1}, \quad k_4 = \frac{1}{2\sqrt{3}}, \quad (38)$$

<sup>15</sup>Note that  $k_2$  is independent of  $k_1$ , in particular its sign. It is usual to set  $k_1 = k_2 = \frac{1}{2\kappa^2} \Rightarrow k_3 = \frac{1}{2}$ .

and<sup>16</sup>

$$(\mathbf{M}_1)^2 = \pm \mathbf{I} , \mathbf{S}_2 = \mathbf{S}_1 \mathbf{M}_1 . \quad (39)$$

In order to completely fulfill the supersymmetric condition  $\delta S = 0$ , it is also proven in Appendix D that the remaining terms of (37) lead to

$$\begin{aligned} 0 = & -\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) \\ & + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) \\ & + \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\ & - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l) , \end{aligned} \quad (40)$$

and

$$\begin{aligned} 0 = & -\frac{i(k_2)^2 \sqrt{\Lambda}}{4k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il} - \mathbf{S}_1^{il} \mathbf{S}_2^{jk} + \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l) \\ & + \frac{i(k_2)^2 \sqrt{\Lambda}}{4k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il} + \mathbf{S}_1^{il} \mathbf{S}_2^{jk} - \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\ & + \frac{i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - 2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} - \mathbf{S}_1^{jl} \mathbf{S}_2^{ik} + \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\ & - \frac{i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - 2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} + \mathbf{S}_1^{jl} \mathbf{S}_2^{ik} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l) . \end{aligned} \quad (41)$$

Each line of the conditions (40)-(41) contains at least one Weyl spinor-ansatz bilinear that vanishes<sup>17</sup> by construction of (20) since it is based on either  $\bar{\epsilon}^i \Psi_\mu^j$ ,  $\bar{\epsilon}^i \gamma_* \Psi_\mu^j$ ,  $\bar{\Psi}_\rho^k \Psi_\sigma^l$  or  $\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l$  with  $i, j$  or  $k, l$  anti-symmetric. *It is important to realize that the conditions (40)-(41) are obeyed only when the Majorana conjugate (5) is used to construct bilinears of Weyl spinors, rather than the Dirac conjugate (4)*<sup>18</sup>.

At this point, we have thus proven that the action (21)-(25) is locally supersymmetric under the local supersymmetry transformations (28)-(29) up to the conditions (39).

## 6 $D = 4, N = 1$ anti-de Sitter supergravity

In this section the plus sign of the cosmological constant term (23) is considered. The following 2x2 matrices satisfy the conditions (39) when it is a plus sign in the first condition:

$$\mathbf{S}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \mathbf{M}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

<sup>16</sup>Note that  $\mathbf{I}$  is the unit matrix.

<sup>17</sup>Taking into account  $(\gamma_*)^2 = \mathbb{1}$  and  $\gamma_*^T C = C \gamma_*$  coming from (3).

<sup>18</sup>An easy way to see this is the fact that  $C$  is block diagonal and  $\gamma^0$  block off-diagonal in the Weyl representation of the  $\gamma$ -matrices given in Appendix A.

$$\mathbf{S}_2 = \mathbf{S}_1 \mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (42)$$

It is easy to see from (20) that  $\psi_\mu \equiv \Psi_\mu^1 + \Psi_\mu^2 = P_L \psi_\mu + P_R(\psi_\mu^C)$  is a Majorana spinor defined by (8). In the case of (42), one can see that the results of the previous section gives back the formulation for  $D = 4, N = 1$  anti-de Sitter supergravity with the gravitino  $\psi_\mu$  expressed as a Majorana spinor<sup>19</sup>:

$$S = k_1 \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}{}^{mn} + 2k_1 \int dx^4 e \Lambda - k_2 \int dx^4 e (\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho) + \frac{k_2 \sqrt{\Lambda}}{\sqrt{3}} \int dx^4 e (\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu), \quad (43)$$

with

$$\begin{aligned} R_{\mu\nu}{}^{mn} &\equiv \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^m{}_r \omega_\nu{}^{rn} - \omega_\nu{}^m{}_r \omega_\mu{}^{rn}, \\ D_\mu \psi_\nu &\equiv \partial_\mu \psi_\nu + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \psi_\nu \implies D_\mu \bar{\psi}_\nu = \partial_\mu \bar{\psi}_\nu - \frac{1}{4} \omega_{\mu mn} \bar{\psi}_\nu \gamma^{mn}, \\ D_{[\rho} e_{\sigma]}^s &= \frac{k_2}{4k_1} \lambda (\bar{\psi}_\rho \gamma^s \psi_\sigma), \\ \omega_\mu{}^{mn} &= \frac{1}{2} e^{m\rho} (\partial_\mu e_\rho^n - \partial_\rho e_\mu^n) - \frac{1}{2} e^{n\rho} (\partial_\mu e_\rho^m - \partial_\rho e_\mu^m) - \frac{1}{2} e^{m\rho} e^{n\sigma} (\partial_\rho e_\sigma^r - \partial_\sigma e_\rho^r) e_{r\mu} + K_\mu{}^{mn}(\psi), \\ K_\mu{}^{mn}(\psi) &= \frac{k_2}{4k_1} \lambda (\bar{\psi}_\mu \gamma^m \psi^n - \bar{\psi}_\mu \gamma^n \psi^m + \bar{\psi}^m \gamma_\mu \psi^n). \end{aligned}$$

and

$$\delta e_\mu^m = \frac{k_2}{2k_1} (\bar{\epsilon} \gamma^m \psi_\mu) \implies \delta e_m^\mu = -\frac{k_2}{2k_1} (\bar{\epsilon} \gamma^\mu \psi_m) \implies \delta e = \frac{k_2}{2k_1} e (\bar{\epsilon} \gamma^\rho \psi_\rho), \quad (44)$$

$$\delta \psi_\mu = D_\mu \epsilon + \frac{\sqrt{\Lambda}}{2\sqrt{3}} \gamma_\mu \epsilon \implies \delta \bar{\psi}_\mu = D_\mu \bar{\epsilon} - \frac{\sqrt{\Lambda}}{2\sqrt{3}} \bar{\epsilon} \gamma_\mu. \quad (45)$$

It is interesting to see that the following 2x2 matrices also satisfy the conditions (39) when it is a plus sign in the first condition:

$$\begin{aligned} \mathbf{S}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{M}_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \\ \mathbf{S}_2 &= \mathbf{S}_1 \mathbf{M}_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \end{aligned} \quad (46)$$

In this case  $\psi_\mu \equiv \Psi_\mu^1 + \Psi_\mu^2 = P_L \psi_\mu + P_R(\psi_\mu^C)$  is still a Majorana spinor but it can no longer be used to derive  $D = 4, N = 1$  anti-de Sitter supergravity. The reason being that the supersymmetry transformation (29) would not give back a Majorana spinor with (46). In our unconventional approach, Weyl spinors play therefore a more important role than Majorana spinors. We will push this result further in the following section.

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<sup>19</sup>See Sec. 2.5 of [3].

## 7 $D = 4, N = 1$ de Sitter supergravity

In this section the minus sign of the cosmological constant term (23) is considered. The following 2x2 matrices satisfy the conditions (39) when it is a minus sign in the first condition:

$$\begin{aligned} \mathbf{S}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{M}_1 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \\ \mathbf{S}_2 &= \mathbf{S}_1 \mathbf{M}_1 = \begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix}, \end{aligned} \quad (47)$$

where  $j$  is the hyperbolic unit<sup>20</sup> having the following properties:

$$\begin{aligned} j^2 &= 1, \\ \bar{j} &= -j \implies j\bar{j} = -1. \end{aligned} \quad (48)$$

It is important to note that the use of the hyperbolic unit does not compromise what has been achieved so far. In particular, the action (21)-(27) remains real and the supersymmetry transformation (29) consistent.

*We have thus proven at this stage that the action (21)-(27) with the matrices (47) is a locally supersymmetric action under (28),(29) for  $D = 4, N = 1$  de Sitter supergravity.*

## 8 Discussion

### 8.1 The use of split-complex numbers in supergravity

Some authors use split-complex numbers<sup>21</sup> in supergravity for the scalar sector [10–13]. *It should not be seen as an issue to use split-complex numbers in supergravity if the action remains real and the supersymmetry transformations consistent.*

### 8.2 Killing spinors

The Killing spinor analysis is similar to the one given in Sec. 2.2.3 of [9]. When  $\Psi_\mu^i = 0$  the bosonic field equations derived from the action (21)-(25) are

$$e_n^\nu R_{\mu\nu}{}^{mn} - \frac{1}{2} e_\mu^m R \mp e_\mu^m \Lambda = 0 \quad (49)$$

whose homogeneous solution is (anti-)de Sitter space with the curvature tensor

$$R_{\mu\nu}{}^{mn} = \mp \frac{\Lambda}{3} (e_\mu^m e_\nu^n - e_\nu^m e_\mu^n). \quad (50)$$

The conditions  $\Psi_\mu^i = 0 \implies \delta\Psi_\mu^i = 0$  lead from (29) to the Killing spinor equation

$$\tilde{D}_\mu \epsilon^i \equiv D_\mu \epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\mu \epsilon^j = 0, \quad (51)$$

<sup>20</sup>The hyperbolic unit is used in mathematics to construct split-complex numbers. See [https://en.wikipedia.org/wiki/Split-complex\\_number](https://en.wikipedia.org/wiki/Split-complex_number) or <https://books.physics.oregonstate.edu/GELG/csplits.html> for instance.

<sup>21</sup>Also called para-complex numbers or hyperbolic numbers in the literature.

whose integrability condition is<sup>22</sup>

$$[\tilde{D}_\mu, \tilde{D}_\nu]\epsilon^i = \left(\frac{1}{4}R_{\mu\nu}{}^{mn}\gamma_{mn} \pm \frac{\Lambda}{6}\gamma_{\mu\nu}\right)\epsilon^i = 0. \quad (52)$$

Substituting (50) into (52), one can see that the integrability condition is obeyed and therefore that Killing spinors do exist even in the de Sitter case, which differs from Section 4.2 of [6].

### 8.3 No extended de Sitter supergravities

One can see that the conditions (40)-(41) are obeyed only when  $i, j, k, l$  range from 1 to 2. Extended de Sitter supergravities are therefore not possible by using our unconventional approach to cancel the quartic fermionic terms appearing in the supersymmetrization process. *This should not be seen as an issue since it is understood that the treatment of the other fundamental interactions—electromagnetic, weak, and strong—differs from that of gravitation.*

## 9 Conclusion

This work shows that it is possible to unconventionally derive  $D = 4, N = 1D$  de Sitter supergravity from a locally supersymmetric action without introducing any additional fields beyond the graviton and gravitino, and without resorting to supersymmetry breaking. The first unconventional feature of this work is the use of the Majorana conjugate to construct bilinears of Weyl spinors in order to cancel the quartic fermionic terms appearing in the supersymmetrization process. The second unconventional feature of this work is the introduction of the hyperbolic unit in the fermionic terms related to a positive cosmological constant.

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<sup>22</sup>This result is calculated in Appendix E.2.

## Appendix

### A Weyl representation of the $\gamma$ -matrices

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ \gamma_* &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \end{aligned} \quad (53)$$

### B Proof that spinor-ansatz bilinears are real

From (2) and (3) we have for any spinor  $\chi^C = B\chi^*$  where  $B = i\gamma^0 C^\dagger \Rightarrow B^{-1} = iC\gamma^0$  with  $(\gamma^0)^2 = -\mathbb{1}$  and  $C^\dagger C = \mathbb{1}$ . Therefore

$$\begin{aligned} \overline{\chi^C} &= (B\chi^*)^T C, \\ &= \chi^\dagger B^T C, \\ &= i\chi^\dagger C^* (\gamma^0)^T C, \\ &\stackrel{(a)}{=} -i\chi^\dagger C^* C \gamma^0, \\ &= -i(\chi^T C)^* C \gamma^0, \\ &= -(\bar{\chi})^* B^{-1}. \end{aligned}$$

The step (a) uses the property  $(\gamma^0)^T C = -C\gamma^0$  coming from (3).

Let's consider the spinor-ansatz bilinear  $M^{ij}\bar{\epsilon}^i N\Psi^j$  based on any two spinor-ansatzes  $\epsilon^i$ ,  $\Psi^i$  defined by (17) or (20), and where  $N$  is any matrix obtained from products of  $\gamma^m$  and  $i\gamma_*$ . From the properties (3) and  $(\gamma^m)^\dagger = \gamma^0 \gamma^m \gamma^0$  with  $(\gamma^0)^2 = -\mathbb{1}$ , it can be shown that  $BN^*B^{-1} = N$ . Hence<sup>23</sup>

$$\begin{aligned} (M^{ij}\bar{\epsilon}^i N\Psi^j)^* &= -(M^{ij})^* (\epsilon^i)^* N^* (\Psi^j)^*, \\ &= -(M^{ij})^* (\bar{\epsilon}^i)^* B^{-1} B N^* B^{-1} B (\Psi^j)^*, \\ &\stackrel{(a)}{=} (M^{ij})^* (\bar{\epsilon}^i)^C N (\Psi^j)^C, \\ &= (M^{11})^* (\bar{\epsilon}^1)^C N (\Psi^1)^C + (M^{12})^* (\bar{\epsilon}^1)^C N (\Psi^2)^C \\ &\quad + (M^{21})^* (\bar{\epsilon}^2)^C N (\Psi^1)^C + (M^{22})^* (\bar{\epsilon}^2)^C N (\Psi^2)^C, \\ &= (M^{11})^* \bar{\epsilon}^2 N \Psi^2 + (M^{12})^* \bar{\epsilon}^2 N \Psi^1 + (M^{21})^* \bar{\epsilon}^1 N \Psi^2 + (M^{22})^* \bar{\epsilon}^1 N \Psi^1, \\ &\stackrel{(b)}{=} M^{22} \bar{\epsilon}^2 N \Psi^2 + M^{21} \bar{\epsilon}^2 N \Psi^1 + M^{12} \bar{\epsilon}^1 N \Psi^2 + M^{11} \bar{\epsilon}^1 N \Psi^1, \\ &\stackrel{(!)}{=} M^{ij} \bar{\epsilon}^i N \Psi^j, \end{aligned}$$

The step (a) uses the above mentioned results  $\overline{\chi^C} = -(\bar{\chi})^* B^{-1}$  and  $BN^*B^{-1} = N$ .

The step (b) uses (19).

<sup>23</sup>As in Sec. 3.2.4 of [4], the convention is used to reverse the order of anti-commuting Grassmann variables (spinor components) in the process of complex conjugation:  $(\alpha\beta)^* = \beta^* \alpha^* = -\alpha^* \beta^*$  where  $\alpha, \beta$  are two anti-commuting Grassmann variables.

## C consistency

### C.1 Consistency of the construction for Weyl spinor-ansatzes (20)

Let's prove the expression for  $\bar{\Psi}^1$ . Since  $\Psi^1 \equiv P_L \psi$ , we have

$$\begin{aligned}
\bar{\Psi}^1 &= \overline{(P_L \psi)}, \\
&\stackrel{(a)}{=} (P_L \psi)^T C, \\
&\stackrel{(b)}{=} \frac{1}{2} [(\mathbb{1} - \gamma_*) \psi]^T C, \\
&= \frac{1}{2} (\psi)^T C - \frac{1}{2} (\psi)^T (\gamma_*)^T C, \\
&\stackrel{(c)}{=} \frac{1}{2} (\psi)^T C - \frac{1}{2} (\psi)^T C \gamma_*, \\
&= (\psi)^T C \frac{1}{2} (\mathbb{1} - \gamma_*), \\
&\stackrel{(d)}{=} \bar{\psi} \frac{1}{2} (\mathbb{1} - \gamma_*), \\
&= P_L \bar{\psi}.
\end{aligned}$$

The step (a) uses the definition (5)

The step (b) uses the definition (10).

The step (c) uses the property  $(\gamma_*)^T C = C \gamma_*$  coming from (3).

The step (d) uses the definition (5).

Let's also prove the expression for  $\Psi^2$ . Since  $\Psi^2 \equiv (\Psi^1)^C$  with  $\Psi^1 \equiv P_L \psi$ , we have

$$\begin{aligned}
\Psi^2 &= (P_L \psi)^C, \\
&\stackrel{(a)}{=} \left[ \frac{1}{2} (\mathbb{1} - \gamma_*) \psi \right]^C, \\
&\stackrel{(b)}{=} \frac{i}{2} \gamma^0 C^\dagger [(\mathbb{1} - \gamma_*) \psi]^*, \\
&= \frac{i}{2} \gamma^0 C^\dagger (\psi)^* - \frac{i}{2} \gamma^0 C^\dagger (\gamma_*)^* (\psi)^*, \\
&\stackrel{(c)}{=} \frac{i}{2} \gamma^0 C^\dagger (\psi)^* - \frac{i}{2} \gamma^0 \gamma_* C^\dagger (\psi)^*, \\
&\stackrel{(d)}{=} \frac{i}{2} \gamma^0 C^\dagger (\psi)^* + \frac{i}{2} \gamma_* \gamma^0 C^\dagger (\psi)^*, \\
&\stackrel{(e)}{=} \frac{1}{2} (\psi)^C + \frac{1}{2} \gamma_* (\psi)^C, \\
&= \frac{1}{2} (\mathbb{1} + \gamma_*) (\psi)^C, \\
&= P_R \psi^C.
\end{aligned}$$

The step (a) uses the definition (10).

The step (b) uses the definition (2).

The step (c) uses the property  $C^\dagger (\gamma_*)^* = \gamma_* C^\dagger$  coming from (3) with  $(\gamma_*)^\dagger = \gamma_*$ .

The step (d) uses the relation  $\gamma^0 \gamma_* = -\gamma_* \gamma^0$ .

The step (e) uses the definition (2).

Let's finally prove the expression for  $\bar{\Psi}^2$ . Since  $\Psi^2 = P_R \psi^C$  from above, we have

$$\begin{aligned}
\bar{\Psi}^1 &= \overline{(P_R \psi^C)}, \\
&\stackrel{(a)}{=} (P_R \psi^C)^T C, \\
&\stackrel{(b)}{=} \frac{1}{2} [(\mathbb{1} + \gamma_*) \psi^C]^T C, \\
&= \frac{1}{2} (\psi^C)^T C + \frac{1}{2} (\psi^C)^T (\gamma_*)^T C, \\
&\stackrel{(c)}{=} \frac{1}{2} (\psi^C)^T C + \frac{1}{2} (\psi^C)^T C \gamma_*, \\
&= (\psi^C)^T C \frac{1}{2} (\mathbb{1} + \gamma_*), \\
&\stackrel{(d)}{=} \overline{(\psi^C)} \frac{1}{2} (\mathbb{1} + \gamma_*), \\
&= P_R \overline{(\psi^C)}.
\end{aligned}$$

The step (a) uses the definition (5).

The step (b) uses the definition (10).

The step (c) uses the property  $(\gamma_*)^T C = C \gamma_*$  coming from (3).

The step (d) uses the definition (5).

## C.2 Consistency of the local supersymmetry transformation (29)

Let's prove the expression for  $\delta \Psi_\mu^2$  by taking into account (13) and the fact that  $\mathbf{M}_1^{ij}$  is block off-diagonal. We have  $\delta \Psi_\mu^1 = D_\mu \varepsilon^1 + k_4 \sqrt{\Lambda} \mathbf{M}_1^{1h} \gamma_\mu \varepsilon^h = \partial_\mu \varepsilon^1 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \varepsilon^1 + k_4 \sqrt{\Lambda} \mathbf{M}_1^{12} \gamma_\mu \varepsilon^2$ . Hence

$$\begin{aligned}
\delta \Psi_\mu^2 &= (\delta \Psi_\mu^1)^C, \\
&= (\partial_\mu \varepsilon^1 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \varepsilon^1 + k_4 \sqrt{\Lambda} \mathbf{M}_1^{12} \gamma_\mu \varepsilon^2)^C, \\
&\stackrel{(a)}{=} B (\partial_\mu \varepsilon^1 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \varepsilon^1 + k_4 \sqrt{\Lambda} \mathbf{M}_1^{12} \gamma_\mu \varepsilon^2)^*, \\
&= B (\partial_\mu \varepsilon^1)^* + \frac{1}{4} B (\omega_{\mu ab} \gamma^{ab} \varepsilon^1)^* + B (k_4 \sqrt{\Lambda} \mathbf{M}_1^{12} \gamma_\mu \varepsilon^2)^*, \\
&= B \partial_\mu (\varepsilon^1)^* + \frac{1}{4} B \omega_{\mu ab} (\gamma^{ab})^* (\varepsilon^1)^* + B k_4 \sqrt{\Lambda} (\mathbf{M}_1^{12})^* (\gamma_\mu)^* (\varepsilon^2)^*, \\
&= \partial_\mu [B (\varepsilon^1)^*] + \frac{1}{4} \omega_{\mu ab} B (\gamma^{ab})^* B^{-1} B (\varepsilon^1)^* + k_4 \sqrt{\Lambda} (\mathbf{M}_1^{12})^* B (\gamma_\mu)^* B^{-1} B (\varepsilon^2)^*, \\
&\stackrel{(b)}{=} \partial_\mu [B (\varepsilon^1)^*] + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} B (\varepsilon^1)^* + k_4 \sqrt{\Lambda} (\mathbf{M}_1^{12})^* \gamma_\mu B (\varepsilon^2)^*, \\
&\stackrel{(c)}{=} \partial_\mu (\varepsilon^1)^C + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} (\varepsilon^1)^C + k_4 \sqrt{\Lambda} (\mathbf{M}_1^{12})^* \gamma_\mu (\varepsilon^2)^C, \\
&\stackrel{(d)}{=} \partial_\mu \varepsilon^2 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \varepsilon^2 + k_4 \sqrt{\Lambda} (\mathbf{M}_1^{12})^* \gamma_\mu \varepsilon^1, \\
&\stackrel{(e)}{=} \partial_\mu \varepsilon^2 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \varepsilon^2 + k_4 \sqrt{\Lambda} \mathbf{M}_1^{21} \gamma_\mu \varepsilon^1, \\
&= D_\mu \varepsilon^2 + k_4 \sqrt{\Lambda} \mathbf{M}_1^{2h} \gamma_\mu \varepsilon^h.
\end{aligned}$$

The step (a) uses the definition (2) with  $B = i\gamma^0 C^\dagger$ .

The step (b) uses the property<sup>24</sup>  $BN^*B^{-1} = N$  where  $N$  is any matrix obtained from products of  $\gamma^m$  and  $i\gamma_*$ .

The step (c) uses the definition (2) with  $B = i\gamma^0 C^\dagger$ .

The step (d) uses (17) or (20).

The step (e) uses (19).

The verification of the expressions for  $\delta\bar{\Psi}_\mu^1$  and  $\delta\bar{\Psi}_\mu^2$  is let to the reader. These expressions follow directly from the definition (5) and the properties  $(\gamma^{ab})^T C = -C\gamma^{ab}$ ,  $(\gamma_\mu)^T C = -C\gamma_\mu$  coming from (3).

## D Detailed calculations

The calculations are cumbersome but not complicated. They require to master Levi-Civita transformations, spinor-ansatz flips, partial integration, supersymmetry transformations and spinor-ansatz reorderings.

Levi-Civita transformations simplify the calculations since they are in most cases easier to perform on expressions that are based on the totally antisymmetric Levi-Civita symbol  $\epsilon^{\mu\nu\rho\sigma} \equiv e\epsilon^{mnr s} e_m^\mu e_n^\nu e_r^\rho e_s^\sigma \Rightarrow \epsilon_{\mu\nu\rho\sigma} = e^{-1}\epsilon_{mnr s} e_\mu^m e_\nu^n e_\rho^r e_\sigma^s$  with  $\epsilon_{0123} = +1 = -\epsilon^{0123}$  and  $e = \det e_\mu^m$ . To do so, the following relations are needed

$$\begin{aligned} e\mathbb{1} &= -\frac{i}{24}\epsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_{\mu\nu\rho\sigma} \iff i\gamma_* = \frac{1}{24}\epsilon_{mnr s}\gamma^{mnr s}, \\ e\gamma^\mu &= \frac{i}{6}\epsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_{\nu\rho\sigma} \iff i\gamma_*\gamma_m = -\frac{1}{6}\epsilon_{mnr s}\gamma^{nrs}, \\ e\gamma^{\mu\nu} &= \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_{\rho\sigma} \iff i\gamma_*\gamma_{mn} = -\frac{1}{2}\epsilon_{mnr s}\gamma^{rs}, \\ e\gamma^{\mu\nu\rho} &= -i\epsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma \iff i\gamma_*\gamma_{mnr} = \epsilon_{mnr s}\gamma^s, \\ e\gamma^{\mu\nu\rho\sigma} &= -i\epsilon^{\mu\nu\rho\sigma}\gamma_* \iff i\gamma_*\gamma_{mnr s} = \epsilon_{mnr s}\mathbb{1}, \end{aligned}$$

where  $\mathbb{1}$  is the identity matrix in the spinor space.

Spinor-ansatz flips are performed by taking the transpose of spinor-ansatz bilinears, by taking into account the properties (3) and by incorporating a minus sign obtained by changing the order of the spinor components that are anti-commuting Grassmann variables. Let's give an example:

$$\begin{aligned} (\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j) &= (\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j)^T, \\ &\stackrel{(a)}{=} [(\Psi_\rho^i)^T C \gamma_* \gamma_n \Psi_\sigma^j]^T, \\ &\stackrel{(b)}{=} -[(\Psi_\sigma^j)^T (\gamma_n)^T (\gamma_*)^T C^T \Psi_\rho^i], \\ &\stackrel{(c)}{=} -[(\Psi_\sigma^j)^T (\gamma_n)^T C^T \gamma_* \Psi_\rho^i], \\ &\stackrel{(d)}{=} -[(\Psi_\sigma^j)^T C \gamma_n \gamma_* \Psi_\rho^i], \end{aligned}$$

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<sup>24</sup>See Appendix B for further details.

$$\stackrel{(e)}{=} [(\bar{\Psi}_\sigma^j)^T C \gamma_* \gamma_n \Psi_\mu^i],$$

$$\stackrel{(f)}{=} (\bar{\Psi}_\sigma^j \gamma_* \gamma_n \Psi_\rho^i).$$

The step (a) uses the definition (5).

The step (b) incorporates a minus sign obtained by changing the order of the spinor component that are anti-commuting Grassmann variables

The step (c) uses the property  $(\gamma_*)^T C^T = C^T \gamma_*$  coming from (3).

The step (d) uses the property  $(\gamma_n)^T C^T = C^T \gamma_n$  coming from (3).

The step (e) uses the relation  $\gamma_n \gamma_* = -\gamma_* \gamma_n$ .

The step (f) uses the definition (5).

Partial integration is the cornerstone of the least action principle. It is performed on the local Lorentz derivative  $D_\mu$  by taking into account that the local Lorentz derivative of a scalar is its partial derivative and that the local Lorentz derivative acts distributively:  $\partial_\mu(\mathbf{S}_2^{ij} \bar{\Psi}_\nu^i \gamma^{\nu\rho} \Psi_\rho^j) = D_\mu(\mathbf{S}_2^{ij} \bar{\Psi}_\nu^i \gamma^{\nu\rho} \Psi_\rho^j) = \mathbf{S}_2^{ij} D_\mu \bar{\Psi}_\nu^i \gamma^{\nu\rho} \Psi_\rho^j + \mathbf{S}_2^{ij} \bar{\Psi}_\nu^i D_\mu \gamma^{\nu\rho} \Psi_\rho^j + \mathbf{S}_2^{ij} \bar{\Psi}_\nu^i \gamma^{\nu\rho} D_\mu \Psi_\rho^j$ . Moreover, since  $\epsilon_{mnr s}, \epsilon^{\mu\nu\rho\sigma}$  are numbers<sup>25</sup>, one has  $D_\mu \epsilon_{mnr s} = 0$  and  $D_\mu \epsilon^{\nu\rho\sigma\alpha} = 0$ . Finally, since  $D_\mu \gamma^m = 0$  and  $D_\mu \gamma_* = 0$  from (14)-(15), one has  $D_\mu \gamma^\nu = \gamma^m D_\mu e_m^\nu$  and  $D_\mu(\gamma_* \gamma^\nu) = \gamma_* \gamma^m D_\mu e_m^\nu$ .

Supersymmetry transformations are performed by applying (28)-(29) when needed in the course of the calculations.

Spinor-ansatz reorderings allow the interchange of spinor-ansatzes between two spinor-ansatz bilinears. They are performed using the fundamental Fierz rearrangement identity

$$\begin{aligned} (\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\ &\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\ &\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4), \end{aligned}$$

where  $\chi_1, \chi_2, \chi_3,$  and  $\chi_4$  are any four spinors.

*These five operations are respectively indicated by L, F, P, S, R placed above the equal sign in the course of the calculations.*

The calculations also require specific relations coming from the algebra of  $\gamma$ -matrices that follow from their defining relation  $\gamma^m \gamma^n + \gamma^n \gamma^m = 2\eta^{mn} \mathbf{1}$ . These relations are specified when needed in the course of the calculations.

Finally, changes of indices are performed throughout the calculations on Lorentz (latin) indices, spacetime (greek) indices and matrix indices. These changes of indices are allowed since the calculations are performed on expressions with contracted indices.

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<sup>25</sup>See Sec. 7.5 of [4].

## D.1 Derivation with respect to the spin connection $\omega_\mu^{mn}$

From (22) we have

$$\begin{aligned}
\delta_\omega S_{\text{EH}} &= +k_1 \delta_\omega \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}{}^{mn}, \\
&\stackrel{(a)}{=} +\frac{k_1}{2} \delta_\omega \int dx^4 e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) R_{\mu\nu}{}^{mn}, \\
&\stackrel{(b)}{=} -\frac{k_1}{4} \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} e_\rho^r e_\sigma^s, \\
&\stackrel{(c)}{=} -\frac{k_1}{4} \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} (\partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^m{}_k \omega_\nu{}^{kn} - \omega_\nu{}^m{}_k \omega_\mu{}^{kn}) e_\rho^r e_\sigma^s, \\
&= -\frac{k_1}{2} \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} (\partial_\mu \omega_\nu{}^{mn} + \omega_\mu{}^m{}_k \omega_\nu{}^{kn}) e_\rho^r e_\sigma^s, \\
&= -\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} (\partial_\mu \delta\omega_\nu{}^{mn} + \omega_\mu{}^m{}_k \delta\omega_\nu{}^{kn} + \omega_\mu{}^n{}_k \delta\omega_\nu{}^{mk}) e_\rho^r e_\sigma^s, \\
&\stackrel{(d)}{=} -\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} D_\mu \delta\omega_\nu{}^{mn} e_\rho^r e_\sigma^s, \\
&\stackrel{P}{=} +\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} \delta\omega_\nu{}^{mn} D_\mu e_\rho^r e_\sigma^s + \frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} \delta\omega_\nu{}^{mn} e_\rho^r D_\mu e_\sigma^s, \\
&= +k_1 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} \delta\omega_\mu{}^{mn} e_\nu^r D_\rho e_\sigma^s.
\end{aligned}$$

The step (a) uses the property  $R_{\mu\nu}{}^{mn} = -R_{\mu\nu}{}^{nm}$  of (26).

The step (b) uses the relation  $e(e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} e_\rho^r e_\sigma^s$ .

The step (c) uses (26).

The step (d) uses the definition of the local Lorentz derivative of the tensor<sup>26</sup>  $\delta\omega_\nu{}^{mn}$  given by  $D_\mu \delta\omega_\nu{}^{mn} \equiv \partial_\mu \delta\omega_\nu{}^{mn} + \omega_\mu{}^m{}_k \delta\omega_\nu{}^{kn} + \omega_\mu{}^n{}_k \delta\omega_\nu{}^{mk}$ .

From (24) we have

$$\begin{aligned}
\delta_\omega S_{\text{RS}} &= -k_2 \delta_\omega \int dx^4 e \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \\
&\stackrel{L}{=} +ik_2 \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\sigma D_\nu \Psi_\rho^j), \\
&\stackrel{(a)}{=} +ik_2 \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} [\bar{\Psi}_\mu^i \gamma_* \gamma_\sigma (\partial_\nu \Psi_\rho^j + \frac{1}{4} \omega_{\nu mn} \gamma^{mn} \Psi_\rho^j)], \\
&= +\frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \delta\omega_{\nu mn} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\sigma \gamma^{mn} \Psi_\rho^j), \\
&\stackrel{C}{=} -\frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \delta\omega_\mu{}^{mn} \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_\nu \gamma_{mn} \Psi_\sigma^j), \\
&\stackrel{(b)}{=} -\frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \delta\omega_\mu{}^{mn} \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_{\nu mn} \Psi_\sigma^j) - \frac{ik_2}{2} \int dx^4 \delta\omega_\mu{}^{mn} e_{m\nu} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j), \\
&\stackrel{LC}{=} -\frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} \delta\omega_\mu{}^{mn} e_\nu^r \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma^s \Psi_\sigma^j).
\end{aligned}$$

The step (a) uses (27).

<sup>26</sup>Note that  $\delta\omega_\nu{}^{mn}$  is a tensor despite the fact that  $\omega_\nu{}^{mn}$  is not.

The step (b) uses the relation  $\gamma_\nu \gamma_{mn} = \gamma_{\nu mn} + e_{m\nu} \gamma_n - e_{n\nu} \gamma_m$ .

The expression  $\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij}(\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j)$  vanishes in (b) since  $\mathbf{S}_1^{ij}(\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j) \stackrel{F}{=} \mathbf{S}_1^{ij}(\bar{\Psi}_\sigma^j \gamma_* \gamma_n \Psi_\rho^i) = \mathbf{S}_1^{ij}(\bar{\Psi}_\sigma^i \gamma_* \gamma_n \Psi_\rho^j)$  is symmetric in  $\rho\sigma$ , which clashes with the total antisymmetry of  $\epsilon^{\mu\nu\rho\sigma}$ .

## D.2 Derivation with respect to the graviton $e_\mu^m$

From (22) we have

$$\begin{aligned}
\delta_e S_{\text{EH}} &= +k_1 \delta_e \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}{}^{mn}, \\
&\stackrel{(a)}{=} +\frac{k_1}{2} \delta_e \int dx^4 e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) R_{\mu\nu}{}^{mn}, \\
&\stackrel{(b)}{=} -\frac{k_1}{4} \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} e_\rho^r e_\sigma^s, \\
&= -\frac{k_1}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} \delta e_\rho^r e_\sigma^s - \frac{k_1}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} e_\rho^r \delta e_\sigma^s, \\
&\stackrel{(c)}{=} -\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} e_\rho^r \delta e_\sigma^s, \\
&\stackrel{S}{=} -\frac{k_1 k_3}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} e_\rho^r \mathbf{S}_1^{ij}(\bar{\epsilon}^i \gamma^s \Psi_\sigma^j).
\end{aligned}$$

The step (a) uses the properties  $R_{\mu\nu}{}^{mn} = -R_{\nu\mu}{}^{mn}$  and  $R_{\mu\nu}{}^{mn} = -R_{\mu\nu}{}^{nm}$  of (26).

The step (b) uses the relation  $e(e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} e_\rho^r e_\sigma^s$ .

Note that the step (c) can also be expressed in the usual form  $-\frac{1}{2} \int dx^4 e (e_n^\nu R_{\mu\nu}{}^{mn} - \frac{1}{2} e_\mu^m R) \delta e_\mu^m$  of general relativity by using the relation  $\epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} e_\sigma^s = -e(e_m^\mu e_n^\nu e_r^\rho - e_n^\mu e_m^\nu e_r^\rho + e_r^\mu e_m^\nu e_n^\rho - e_r^\mu e_n^\nu e_m^\rho + e_n^\mu e_r^\nu e_m^\rho - e_m^\mu e_r^\nu e_n^\rho)$ .

From (23) we have

$$\begin{aligned}
\delta_e S_\Lambda &= \pm 2k_1 \delta_e \int dx^4 e, \\
&= \pm 2k_1 \int dx^4 \delta e, \\
&\stackrel{S}{=} \pm 2k_1 k_3 \int dx^4 e \mathbf{S}_1^{ij}(\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j).
\end{aligned}$$

From (24) we have

$$\begin{aligned}
\delta_e S_{\text{RS}} &= -k_2 \delta_e \int dx^4 e \mathbf{S}_1^{ij}(\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \\
&\stackrel{L}{=} +ik_2 \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij}(\bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j), \\
&= +ik_2 \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij}(\bar{\Psi}_\mu^i \gamma_* \gamma_m D_\rho \Psi_\sigma^j) e_\nu^m, \\
&= -ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij}(\bar{\Psi}_\nu^i \gamma_* \gamma_m D_\rho \Psi_\sigma^j) \delta e_\mu^m, \\
&\stackrel{S}{=} -ik_2 k_3 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij}(\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_1^{kl}(\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l).
\end{aligned}$$

From (25) we have

$$\begin{aligned}
\delta_e S_{g\Lambda} &= +2k_2 k_4 \sqrt{\Lambda} \delta_e \int dx^4 e \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu} \Psi_\nu^j), \\
&\stackrel{L}{=} +ik_2 k_4 \sqrt{\Lambda} \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\rho \Psi_\nu^j), \\
&= +ik_2 k_4 \sqrt{\Lambda} \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) e_\rho^m e_\sigma^n, \\
&= +ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) \delta e_\rho^m e_\sigma^n + ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) e_\rho^m \delta e_\sigma^n, \\
&= +2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) e_\rho^m \delta e_\sigma^n, \\
&\stackrel{S}{=} -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma_{m\rho} \Psi_\sigma^l), \\
&\stackrel{(a)}{=} -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) \\
&\quad + 2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l), \\
&\stackrel{(a)}{=} -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the relation  $\gamma_{m\rho} = \gamma_m \gamma_\rho - e_{m\rho}$ .

### D.3 Derivation with respect to the gravitino $\Psi_\mu^i$

From (24) we have

$$\begin{aligned}
\delta_\Psi S_{RS} &= -k_2 \delta_\Psi \int dx^4 e \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \\
&\stackrel{L}{=} +ik_2 \delta_\Psi \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j), \\
&= +ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) + ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \delta \Psi_\sigma^j), \\
&\stackrel{P}{=} +ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (D_\mu \bar{\Psi}_\nu^i \gamma_* \gamma_\rho \delta \Psi_\sigma^j) \\
&\quad + ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* D_\nu \gamma_\rho \delta \Psi_\sigma^j), \\
&\stackrel{F(a)}{=} +2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{S}{=} +2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} ([D_\mu \bar{\epsilon}^i - k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \bar{\epsilon}^h \gamma_\mu] \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} ([D_\mu \bar{\epsilon}^i - k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \bar{\epsilon}^h \gamma_\mu] \gamma_* D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{(b)}{=} +2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (D_\mu \bar{\epsilon}^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (D_\mu \bar{\epsilon}^i \gamma_* D_\nu \gamma_\rho \Psi_\sigma^j) - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
&\stackrel{P}{=} -2ik_2 \int dx^4 \overbrace{\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j)}^a + 2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* \gamma_\mu D_\nu D_\rho \Psi_\sigma^j) \\
&\quad + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_\nu D_\rho \Psi_\sigma^j) + ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* D_\mu D_\nu \gamma_\rho \Psi_\sigma^j) \\
&\quad + ik_2 \int dx^4 \overbrace{\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j)}^a - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{(c)}{=} + \frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* \gamma_{mnr} \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_{\mu\nu} D_\rho \Psi_\sigma^j) + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \eta_{\mu\nu} D_\rho \Psi_\sigma^j) \\
&\quad - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{L}{=} + \frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^s \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad + 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{(d)}{=} + \frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnr s} R_{\mu\nu}{}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma^s \Psi_\sigma^j) + 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) \\
&\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l).
\end{aligned}$$

The step (a) justifies the presupposed symmetries  $\mathbf{S}_1^T = \mathbf{S}_1$ .

The step (b) uses the relations  $\gamma_\mu \gamma_* = -\gamma_* \gamma_\mu$  and  $\gamma_\mu \gamma^{\alpha\beta} \gamma_* = -\gamma_* \gamma_\mu \gamma^{\alpha\beta}$ .

The step (c) uses the results<sup>27</sup>  $[D_\mu, D_\nu] \Psi_\rho^i = \frac{1}{4} R_{\mu\nu}{}^{mn} \gamma_{mn} \Psi_\rho^i$  and  $[D_\mu, D_\nu] \gamma_\rho = R_{\mu\nu}{}^{mn} \gamma_m e_{n\rho}$ . Therefore one can see that  $i \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\nu D_\mu D_\rho \Psi_\sigma^i) + \frac{i}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* D_\mu D_\nu \gamma_\rho \Psi_\sigma^i) = \frac{i}{8} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} (\bar{\epsilon}^i \gamma_* \gamma_\rho \gamma_{mn} \Psi_\sigma^i) + \frac{i}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\sigma^i) e_{n\rho} = \frac{i}{8} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} e_\rho^r (\bar{\epsilon}^i \gamma_* \gamma_{mnr} \Psi_\sigma^i)$  with the relation  $\gamma_r \gamma_{mn} - e_{mr} \gamma_n + e_{nr} \gamma_m = \gamma_{r mn}$ .

The step (c) also uses the relation  $\gamma_\mu \gamma_\nu = \gamma_{\mu\nu} + \eta_{\mu\nu}$ .

The expression  $\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_* \eta_{\mu\nu} D_\rho \Psi_\sigma^j)$  vanishes in (c) since  $\eta_{\mu\nu}$  is symmetric in  $\mu\nu$ , which clashes with the total antisymmetry of  $\epsilon^{\mu\nu\rho\sigma}$ .

The step (d) uses the result  $D_{[\rho} e_{\sigma]}^m = \frac{k_2}{4k_1} \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma^m \Psi_\sigma^j)$ .

From (25) we have

$$\begin{aligned}
\delta_\Psi S_{g\Lambda} &= +2k_2 k_4 \sqrt{\Lambda} \delta_\Psi \int dx^4 e \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu} \Psi_\nu^j), \\
&\stackrel{L}{=} + ik_2 k_4 \sqrt{\Lambda} \delta_\Psi \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\nu \rho \Psi_\sigma^j), \\
&= + ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu \rho \Psi_\sigma^j) + ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\nu \rho \delta \Psi_\sigma^j), \\
&\stackrel{F}{=} + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu \rho \Psi_\sigma^j),
\end{aligned}$$

<sup>27</sup>These are calculated in Appendix E.1.

$$\begin{aligned}
&\stackrel{S}{=} +2ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} ([D_\mu \bar{\epsilon}^i - k_4\sqrt{\Lambda} \mathbf{M}_1^{ih} \bar{\epsilon}^h \gamma_\mu] \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&= +2ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (D_\mu \bar{\epsilon}^i \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j) - 2ik_2(k_4)^2 \Lambda \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_\mu \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&\stackrel{P}{=} -2ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* D_\mu \gamma_{\nu\rho} \Psi_\sigma^j) - 2ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \gamma_{\mu\nu} D_\rho \Psi_\sigma^j) \\
&\quad - 2ik_2(k_4)^2 \Lambda \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_\mu \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&\stackrel{L}{=} -2ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* D_\mu \gamma_{\nu\rho} \Psi_\sigma^j) - 4k_2k_4\sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 4k_2(k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma_\mu \gamma^{\mu\nu} \Psi_\nu^j), \\
&\stackrel{(a)}{=} -2ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* D_\mu \gamma_{\nu\rho} \Psi_\sigma^j) - 4k_2k_4\sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2(k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j), \\
&\stackrel{(b)}{=} -2ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \gamma_{mn} \Psi_\sigma^j) (D_\mu e_\nu^m e_\rho^n + e_\nu^m D_\mu e_\rho^n) - 4k_2k_4\sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2(k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j), \\
&= +4ik_2k_4\sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \gamma_{\mu m} \Psi_\nu^j) D_\rho e_\sigma^m - 4k_2k_4\sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2(k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j), \\
&\stackrel{(c)}{=} + \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \gamma_{\mu m} \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) - 4k_2k_4\sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2(k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j), \\
&\stackrel{(d)L}{=} -4k_2k_4\sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) - 12k_2(k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j) \\
&\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\
&\quad + \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the relations  $\gamma_\mu \gamma^{\mu\nu} = 3\gamma^\nu$  and  $\gamma_\mu \gamma^{\alpha\beta} \gamma^{\mu\nu} = -\gamma^{\nu\alpha\beta} - g^{\nu\alpha} \gamma^\beta + g^{\nu\beta} \gamma^\alpha$ .

The step (b) uses the definitions  $\gamma_{\nu\rho} = \gamma_{mn} e_\nu^m e_\rho^n$ .

The step (c) uses the result  $D_{[\rho} e_{\sigma]}^m = \frac{k_2}{4k_1} \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma^m \Psi_\sigma^j)$ .

The step (d) uses the relation  $\gamma_{\mu m} = \gamma_\mu \gamma_m - e_{m\mu}$ .

#### D.4 Proof of (40)

$$\begin{aligned}
\delta_e S_{\text{RS}} &\longrightarrow -\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l), \\
\delta_\Psi S_{\text{RS}} &\longrightarrow -\frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l).
\end{aligned}$$

$$\begin{aligned}
0 &\stackrel{?}{=} -\frac{i(k_2)^2}{2k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l)} \\
&\quad -\frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) \\
\stackrel{R(a)}{=} &-\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad + \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j)} \\
&\quad + \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)} \\
&\quad - \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l)}, \\
\stackrel{(b)}{=} &-\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad - \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j)} \\
&\quad + \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)}, \\
\stackrel{R(c)}{=} &-\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) \\
&\quad + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l), \\
\stackrel{(d)}{=} &-\frac{i(k_2)^2}{2k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l)}^c + 2 \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l)}^d \\
&\quad - \frac{i(k_2)^2}{2k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l)}^a \\
&\quad + \frac{i(k_2)^2}{2k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l)}^b, \\
= &-\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) \\
&\quad + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) \\
&\quad - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\
&\quad + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l), \\
\stackrel{(d)}{=} &-\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l)
\end{aligned}$$

$$\begin{aligned}
& + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\
& - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the spinor-ansatz reordering  $\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$  obtained below.

The step (b) uses the spinor-ansatz flips  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j)$  and  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) = \epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$ .

The step (c) uses the spinor-ansatz reordering  $\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) = 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l)$  obtained below.

The step (d) uses the spinor-ansatz flips  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l)$  and  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \Psi_\sigma^l)$ .

### Fierz rearrangement 1

$$\begin{aligned}
(\bar{\chi}_1 \chi_2) (\bar{\chi}_3 \chi_4) &= -\frac{1}{4} (\bar{\chi}_3 \chi_2) (\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2) (\bar{\chi}_1 \gamma_* \chi_4) \\
&- \frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2) (\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2) (\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&+ \frac{1}{8} (\bar{\chi}_3 \gamma_{ab} \chi_2) (\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_* \gamma^m, \chi_2 = D_\rho \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_m \Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l) (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_m \Psi_\mu^j) - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma_m \Psi_\mu^j) \\
&- \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_a D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma^a \gamma_m \Psi_\mu^j) \\
&+ \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_a D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\mu^j) \\
&+ \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_{ab} D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\mu^j), \\
&= +\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&- \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
(1) \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l) &= \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
- \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) &- \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j).
\end{aligned}$$

### Fierz rearrangement 2a

$$(\bar{\chi}_1 \chi_2) (\bar{\chi}_3 \chi_4) = -\frac{1}{4} (\bar{\chi}_3 \chi_2) (\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2) (\bar{\chi}_1 \gamma_* \chi_4)$$

$$\begin{aligned}
& -\frac{1}{4}(\bar{\chi}_3\gamma_a\chi_2)(\bar{\chi}_1\gamma^a\chi_4) + \frac{1}{4}(\bar{\chi}_3\gamma_*\gamma_a\chi_2)(\bar{\chi}_1\gamma_*\gamma^a\chi_4) \\
& + \frac{1}{8}(\bar{\chi}_3\gamma_{ab}\chi_2)(\bar{\chi}_1\gamma^{ab}\chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k\gamma_*, \chi_2 = D_\rho\Psi_\sigma^l, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^k\gamma_*D_\rho\Psi_\sigma^l)(\bar{\varepsilon}^i\Psi_\mu^j) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^iD_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\Psi_\mu^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*D_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_*\Psi_\mu^j) \\
& -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_aD_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma^a\Psi_\mu^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*\gamma_aD_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_*\gamma^a\Psi_\mu^j) \\
& + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_{ab}D_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma^{ab}\Psi_\mu^j) \\
& = +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^iD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\Psi_\sigma^j) \\
& + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^m\Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma^m\Psi_\sigma^j) \\
& - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_{mn}D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^{mn}\Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
(2a) \quad \epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_*D_\rho\Psi_\sigma^l) &= +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^iD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^j) \\
& + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^m\Psi_\sigma^j) \\
& - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma^m\Psi_\sigma^j) - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_{mn}D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^{mn}\Psi_\sigma^j)
\end{aligned}$$

### Fierz rearrangement 2b

$$\begin{aligned}
(\bar{\chi}_1\chi_2)(\bar{\chi}_3\chi_4) &= -\frac{1}{4}(\bar{\chi}_3\chi_2)(\bar{\chi}_1\chi_4) - \frac{1}{4}(\bar{\chi}_3\gamma_*\chi_2)(\bar{\chi}_1\gamma_*\chi_4) \\
& -\frac{1}{4}(\bar{\chi}_3\gamma_a\chi_2)(\bar{\chi}_1\gamma^a\chi_4) + \frac{1}{4}(\bar{\chi}_3\gamma_*\gamma_a\chi_2)(\bar{\chi}_1\gamma_*\gamma^a\chi_4) \\
& + \frac{1}{8}(\bar{\chi}_3\gamma_{ab}\chi_2)(\bar{\chi}_1\gamma^{ab}\chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k, \chi_2 = D_\rho\Psi_\sigma^l, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \gamma_*\Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^kD_\rho\Psi_\sigma^l)(\bar{\varepsilon}^i\gamma_*\Psi_\mu^j) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^iD_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\Psi_\mu^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*D_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_*\Psi_\mu^j) \\
& -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_aD_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma^a\gamma_*\Psi_\mu^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*\gamma_aD_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma^a\gamma_*\Psi_\mu^j) \\
& + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_{ab}D_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma^{ab}\gamma_*\Psi_\mu^j), \\
& = +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^iD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\Psi_\sigma^j) \\
& -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^m\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma^m\Psi_\sigma^j) \\
& - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_{mn}D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^{mn}\Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
(2b) \quad \epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*\Psi_\mu^j)(\bar{\Psi}_\nu^kD_\rho\Psi_\sigma^l) &= +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^iD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^j) \\
& + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^m\Psi_\sigma^j) \\
& + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_*\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma^m\Psi_\sigma^j) - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i\gamma_{mn}D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^{mn}\Psi_\sigma^j).
\end{aligned}$$

$$(2a) - (2b) \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) = +\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$$

$$\Rightarrow (2) \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) = 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l).$$

## D.5 Proof of (41)

$$\begin{aligned} \delta_e S_{g\Lambda} &\longrightarrow -\frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l), \\ \delta_\Psi S_{RS} &\longrightarrow -\frac{i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} \overbrace{(\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l)}^a, \\ \delta_\Psi S_{g\Lambda} &\longrightarrow -\frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\ &\quad + \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} \overbrace{(\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l)}^a. \end{aligned}$$

$$\begin{aligned} 0 &\stackrel{?}{=} -\frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) \\ &\quad + \frac{3i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) \\ &\quad - \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l), \\ &\stackrel{(a)}{=} + \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) - \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) \\ &\quad - \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) + \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l) \\ &\quad + \frac{3i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) \\ &\quad - \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l), \\ &\stackrel{(b)}{=} + \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) - \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) \\ &\quad - \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) + \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l) \\ &\quad - \frac{3i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^l) (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^j) \\ &\quad - \frac{3i(k_2)^2 \sqrt{\Lambda}}{4k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) - \frac{3i(k_2)^2 \sqrt{\Lambda}}{4k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) \\ &\quad - \frac{i(k_2)^2 \sqrt{\Lambda}}{2k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l), \end{aligned}$$



$$\begin{aligned}
& -\frac{i(k_2)^2\sqrt{\Lambda}}{2k_1\sqrt{3}} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ik} \mathbf{S}_1^{jl} (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l)}^a - \frac{i(k_2)^2\sqrt{\Lambda}}{2k_1\sqrt{3}} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ik} \mathbf{S}_1^{jl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l)}^b \\
& -\frac{i(k_2)^2\sqrt{\Lambda}}{2k_1\sqrt{3}} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l)}^c - \frac{i(k_2)^2\sqrt{\Lambda}}{2k_1\sqrt{3}} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l)}^d, \\
= & -\frac{i(k_2)^2\sqrt{\Lambda}}{2k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} + \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2\sqrt{\Lambda}}{2k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\
& - \frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\
& + \frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} + \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l), \\
\stackrel{(f)}{=} & -\frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il} - \mathbf{S}_1^{il} \mathbf{S}_2^{jk} + \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il} + \mathbf{S}_1^{il} \mathbf{S}_2^{jk} - \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2\sqrt{\Lambda}}{8k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - 2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} - \mathbf{S}_1^{jl} \mathbf{S}_2^{ik} + \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\
& - \frac{i(k_2)^2\sqrt{\Lambda}}{8k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - 2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} + \mathbf{S}_1^{jl} \mathbf{S}_2^{ik} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the spinor-ansatz reordering  $\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) = +\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l)$  obtained below.

The step (b) uses the spinor-ansatz reordering  $\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^l) (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^j) + 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j)$  obtained below.

The step (c) uses the spinor-ansatz reordering  $\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l) (\bar{\epsilon}^i \Psi_\mu^j) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k)$  obtained below.

The step (d) uses the spinor-ansatz reordering  $\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) = -2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)$  obtained below.

The step (e) uses the spinor-ansatz reordering  $\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) = +2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k)$  obtained below.

The step (f) uses the spinor-ansatz flips  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^l \gamma_* \gamma_\rho \Psi_\sigma^k)$ ,  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^l \gamma_\rho \Psi_\sigma^k)$ ,  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^l \gamma_* \Psi_\sigma^k)$  and  $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^l \Psi_\sigma^k)$  obtained below.

### Fierz rearrangement 1a

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_* \gamma^m, \chi_2 = \gamma_\rho \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_m \Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l)(\bar{\epsilon}^i \gamma_m \Psi_\mu^j) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_m \Psi_\mu^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma_m \Psi_\mu^j) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_a \gamma_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma^a \gamma_m \Psi_\mu^j) \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_a \gamma_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\mu^j) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_{ab} \gamma_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\mu^j), \\
&= +\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j), \\
&\stackrel{F}{=} +\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k),
\end{aligned}$$

$$\begin{aligned}
(1a) \quad \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \Psi_\mu^j)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) &= \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
+ \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) &- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k).
\end{aligned}$$

### Fierz rearrangement 1b

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^j \gamma_* \gamma^m, \chi_2 = \Psi_\sigma^k, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_m \gamma_\mu \Psi_\nu^l$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k)(\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \Psi_\sigma^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_m \gamma_\mu \Psi_\nu^l) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \Psi_\sigma^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_a \Psi_\sigma^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^a \gamma_m \gamma_\mu \Psi_\nu^l) \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_a \Psi_\sigma^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \gamma_\mu \Psi_\nu^l) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_{ab} \Psi_\sigma^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^{ab} \gamma_m \gamma_\mu \Psi_\nu^l), \\
&= +\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \Psi_\mu^k)(\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \Psi_\mu^k)(\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l)
\end{aligned}$$

$$-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^l) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^l),$$

$$(1b) \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma_*\gamma^m\Psi_\sigma^k) = \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma_\rho\Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_\rho\Psi_\sigma^l) \\ - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^l) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^l)$$

### Fierz rearrangement 1c

$$(\bar{\chi}_1\chi_2)(\bar{\chi}_3\chi_4) = -\frac{1}{4}(\bar{\chi}_3\chi_2)(\bar{\chi}_1\chi_4) - \frac{1}{4}(\bar{\chi}_3\gamma_*\chi_2)(\bar{\chi}_1\gamma_*\chi_4) \\ - \frac{1}{4}(\bar{\chi}_3\gamma_a\chi_2)(\bar{\chi}_1\gamma^a\chi_4) + \frac{1}{4}(\bar{\chi}_3\gamma_*\gamma_a\chi_2)(\bar{\chi}_1\gamma_*\gamma^a\chi_4) \\ + \frac{1}{8}(\bar{\chi}_3\gamma_{ab}\chi_2)(\bar{\chi}_1\gamma^{ab}\chi_4)$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^j\gamma^m, \chi_2 = \Psi_\sigma^k, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_*\gamma_m\gamma_\mu\Psi_\nu^l$$

$$\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\rho^j\gamma^m\Psi_\sigma^k)(\bar{\epsilon}^i\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l) = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\sigma^k)(\bar{\Psi}_\rho^j\gamma^m\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\sigma^k)(\bar{\Psi}_\rho^j\gamma^m\gamma_*\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l) \\ - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_a\Psi_\sigma^k)(\bar{\Psi}_\rho^j\gamma^m\gamma^a\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l) \\ + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_a\Psi_\sigma^k)(\bar{\Psi}_\rho^j\gamma^m\gamma_*\gamma^a\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l) \\ + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_{ab}\Psi_\sigma^k)(\bar{\Psi}_\rho^j\gamma^m\gamma^{ab}\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l), \\ = -\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma_\rho\Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_\rho\Psi_\sigma^l) \\ - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^l) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^l) \\ + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_{ab}\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma^{ab}\gamma_m\gamma_\rho\Psi_\sigma^l),$$

$$(1c) \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma^m\Psi_\sigma^k) = -\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma_\rho\Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_\rho\Psi_\sigma^l) \\ - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^l) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^l)$$

$$(1b)-(1c) \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma_*\gamma^m\Psi_\sigma^k) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma^m\Psi_\sigma^k) = \\ + 2\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma_\rho\Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_\rho\Psi_\sigma^l)$$

$$\Rightarrow (1) \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^l) = +\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^j) \\ - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\Psi_\sigma^j) + \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma_\rho\Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_\rho\Psi_\sigma^l)$$

### Fierz rearrangement 2a

$$(\bar{\chi}_1\chi_2)(\bar{\chi}_3\chi_4) = -\frac{1}{4}(\bar{\chi}_3\chi_2)(\bar{\chi}_1\chi_4) - \frac{1}{4}(\bar{\chi}_3\gamma_*\chi_2)(\bar{\chi}_1\gamma_*\chi_4) \\ - \frac{1}{4}(\bar{\chi}_3\gamma_a\chi_2)(\bar{\chi}_1\gamma^a\chi_4) + \frac{1}{4}(\bar{\chi}_3\gamma_*\gamma_a\chi_2)(\bar{\chi}_1\gamma_*\gamma^a\chi_4) \\ + \frac{1}{8}(\bar{\chi}_3\gamma_{ab}\chi_2)(\bar{\chi}_1\gamma^{ab}\chi_4)$$



### Fierz rearrangement 3a

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_* \gamma_\rho, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l)(\bar{\varepsilon}^i \Psi_\mu^j) &= -\frac{1}{4}(\bar{\varepsilon}^i \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\mu^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma_* \Psi_\mu^j) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_a \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^a \Psi_\mu^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \gamma_a \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma_* \gamma^a \Psi_\mu^j) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_{ab} \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{ab} \Psi_\mu^j), \\
&= +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^j) \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^m \Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_\rho \gamma^m \Psi_\sigma^j) \\
&\quad - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_{mn} \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j), \\
&\stackrel{F}{=} +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^j) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma^m \gamma_\rho \Psi_\sigma^j) \\
&\quad - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_{mn} \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
(3a) \quad \epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l)(\bar{\varepsilon}^i \Psi_\mu^j) &= \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^j) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma^m \gamma_\rho \Psi_\sigma^j) \\
&\quad - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_{mn} \Psi_\mu^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j)
\end{aligned}$$

### Fierz rearrangement 3b

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_\rho, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \gamma_* \Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l)(\bar{\varepsilon}^i \gamma_* \Psi_\mu^j) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_\rho \gamma_* \Psi_\mu^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_\rho \gamma_* \gamma_* \Psi_\mu^j) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_a \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_\rho \gamma^a \gamma_* \Psi_\mu^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_* \gamma_a \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_\rho \gamma_* \gamma^a \gamma_* \Psi_\mu^j) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}^i \gamma_{ab} \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_\rho \gamma^{ab} \gamma_* \Psi_\mu^j),
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^j) \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\gamma^m\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_\rho\gamma^m\Psi_\sigma^j) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_{mn}\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\gamma^{mn}\Psi_\sigma^j), \\
&\stackrel{F}{=} -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^j) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^k) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_{mn}\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\gamma^{mn}\Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
(3b) \quad &\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^l) = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\Psi_\sigma^j) + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^j) \\
&- \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^k) \\
&+ \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_{mn}\Psi_\mu^l)(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\gamma^{mn}\Psi_\sigma^j)
\end{aligned}$$

$$\begin{aligned}
(3a)+(3b) \quad &\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\Psi_\sigma^l)(\bar{\epsilon}^i\Psi_\mu^j) + \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^l) = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k) \\
&- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^k) \\
\Rightarrow (3) \quad &\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\Psi_\sigma^l)(\bar{\epsilon}^i\Psi_\mu^j) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k) \\
&- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^k)
\end{aligned}$$

#### Fierz rearrangement 4a

$$\begin{aligned}
(\bar{\chi}_1\chi_2)(\bar{\chi}_3\chi_4) &= -\frac{1}{4}(\bar{\chi}_3\chi_2)(\bar{\chi}_1\chi_4) - \frac{1}{4}(\bar{\chi}_3\gamma_*\chi_2)(\bar{\chi}_1\gamma_*\chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3\gamma_a\chi_2)(\bar{\chi}_1\gamma^a\chi_4) + \frac{1}{4}(\bar{\chi}_3\gamma_*\gamma_a\chi_2)(\bar{\chi}_1\gamma^*\gamma^a\chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3\gamma_{ab}\chi_2)(\bar{\chi}_1\gamma^{ab}\chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^j\gamma_*\gamma^m, \chi_2 = \gamma_\rho\Psi_\sigma^k, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_m\Psi_\mu^l$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k)(\bar{\epsilon}^i\gamma_m\Psi_\mu^l) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_\rho\Psi_\sigma^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_m\Psi_\mu^l) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_\rho\Psi_\sigma^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_*\gamma_m\Psi_\mu^l) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_a\gamma_\rho\Psi_\sigma^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma^a\gamma_m\Psi_\mu^l) \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_a\gamma_\rho\Psi_\sigma^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_*\gamma^a\gamma_m\Psi_\mu^l) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_{ab}\gamma_\rho\Psi_\sigma^k)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma^{ab}\gamma_m\Psi_\mu^l), \\
&= +\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\gamma_*\Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\Psi_\sigma^l) \\
&\quad - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\gamma_*\gamma^m\Psi_\sigma^l) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\gamma^m\Psi_\sigma^l),
\end{aligned}$$

$$\begin{aligned}
(4a) \quad &\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k) = \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\gamma_*\Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\Psi_\sigma^l) \\
&- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_m\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\gamma_*\gamma^m\Psi_\sigma^l) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i\gamma_*\gamma_m\gamma_\mu\Psi_\nu^k)(\bar{\Psi}_\rho^j\gamma^m\Psi_\sigma^l)
\end{aligned}$$

### Fierz rearrangement 4b

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^j \gamma^m, \chi_2 = \gamma_\rho \Psi_\sigma^k, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_* \gamma_m \Psi_\mu^l$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k)(\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\rho \Psi_\sigma^k)(\bar{\Psi}_\nu^j \gamma^m \gamma_* \gamma_m \Psi_\mu^l) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\rho \Psi_\sigma^k)(\bar{\Psi}_\nu^j \gamma^m \gamma_* \gamma_* \gamma_m \Psi_\mu^l) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_a \gamma_\rho \Psi_\sigma^k)(\bar{\Psi}_\nu^j \gamma^m \gamma^a \gamma_* \gamma_m \Psi_\mu^l) \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_a \gamma_\rho \Psi_\sigma^k)(\bar{\Psi}_\nu^j \gamma^m \gamma_* \gamma^a \gamma_* \gamma_m \Psi_\mu^l) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_{ab} \gamma_\rho \Psi_\sigma^k)(\bar{\Psi}_\nu^j \gamma^m \gamma^{ab} \gamma_* \gamma_m \Psi_\mu^l), \\
&= -\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \Psi_\sigma^l) \\
&\quad - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) \\
&\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_{ab} \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\sigma^l),
\end{aligned}$$

$$\begin{aligned}
(4b) \quad \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) &= -\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \Psi_\sigma^l) \\
- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) &- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)
\end{aligned}$$

$$\begin{aligned}
(4a)+(4b) \quad \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) &+ \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) = \\
- \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) &- \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)
\end{aligned}$$

$$\begin{aligned}
\gamma_m \gamma_\mu = 2e_{m\mu} - \gamma_\mu \gamma_m \Rightarrow (4) \quad \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) &+ \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l)(\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) = \\
- 2\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \Psi_\mu^k)(\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) &- 2\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \Psi_\mu^k)(\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l) \\
+ \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) &+ \epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k)(\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)
\end{aligned}$$

### Fierz rearrangement 5a

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^j \gamma_* \gamma^m, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i \gamma_\mu, \chi_4 = \gamma_m \Psi_\nu^k$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l)(\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \Psi_\sigma^l)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_m \Psi_\nu^k) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \gamma_* \Psi_\sigma^l)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma_m \Psi_\nu^k) \\
&\quad - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}^i \gamma_\mu \gamma_a \Psi_\sigma^l)(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^a \gamma_m \Psi_\nu^k)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\nu^k) \\
& + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_{ab} \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\nu^k), \\
= & + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k),
\end{aligned}$$

$$\begin{aligned}
(5a) \quad & \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k)
\end{aligned}$$

**Fierz rearrangement 5b**

$$\begin{aligned}
(\bar{\chi}_1 \chi_2) (\bar{\chi}_3 \chi_4) &= -\frac{1}{4} (\bar{\chi}_3 \chi_2) (\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2) (\bar{\chi}_1 \gamma_* \chi_4) \\
& - \frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2) (\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2) (\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
& + \frac{1}{8} (\bar{\chi}_3 \gamma_{ab} \chi_2) (\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^j \gamma^m, \quad \chi_2 = \Psi_\sigma^l, \quad \bar{\chi}_3 = \bar{\epsilon}^i \gamma_* \gamma_\mu, \quad \chi_4 = \gamma_m \Psi_\nu^k$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma_m \Psi_\nu^k) - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_* \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma_* \gamma_m \Psi_\nu^k) \\
& - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma^a \gamma_m \Psi_\nu^k) \\
& + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\nu^k) \\
& + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_{ab} \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma^{ab} \gamma_m \Psi_\nu^k), \\
= & + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\sigma^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k) + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k),
\end{aligned}$$

$$\begin{aligned}
(5b) \quad & \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) \\
& + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k)
\end{aligned}$$

$$\begin{aligned}
(5a)+(5b) \Rightarrow (5) \quad & \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) = \\
& + 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + 2\epsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k)
\end{aligned}$$

## E Miscellaneous

### E.1 Calculation of $D_{[\mu} D_{\nu]} \Psi_\rho^i$ and $D_{[\mu} D_{\nu]} \gamma_\rho$

Let's first calculate  $D_{[\mu} D_{\nu]} \Psi_\rho^i$

$$\begin{aligned}
D_{[\mu} D_{\nu]} \Psi_\rho^i &= D_\mu D_\nu \Psi_\rho^i - D_\nu D_\mu \Psi_\rho^i, \\
&\stackrel{(a)}{=} D_\mu (\partial_\nu \chi + \frac{1}{4} \omega_{\nu ab} \gamma^{ab} \chi) - D_\nu (\partial_\mu \chi + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \chi),
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(b)}{=} \partial_\mu(\partial_\nu\chi + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\chi) + \frac{1}{4}\omega_{\mu cd}\gamma^{cd}(\partial_\nu\chi + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\chi) \\
&\quad - \partial_\nu(\partial_\mu\chi - \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\chi) - \frac{1}{4}\omega_{\nu cd}\gamma^{cd}(\partial_\mu\chi - \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\chi), \\
&= \cancel{\partial_\mu\partial_\nu\chi}^a + \frac{1}{4}\partial_\mu\omega_{\nu ab}\gamma^{ab}\chi + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\cancel{\partial_\mu\chi}^b + \frac{1}{4}\omega_{\mu cd}\gamma^{cd}\cancel{\partial_\nu\chi}^c + \frac{1}{16}\omega_{\mu cd}\omega_{\nu ab}\gamma^{cd}\gamma^{ab}\chi \\
&\quad - \cancel{\partial_\nu\partial_\mu\chi}^a - \frac{1}{4}\partial_\nu\omega_{\mu ab}\gamma^{ab}\chi - \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\cancel{\partial_\nu\chi}^c - \frac{1}{4}\omega_{\nu cd}\gamma^{cd}\cancel{\partial_\mu\chi}^b - \frac{1}{16}\omega_{\nu cd}\omega_{\mu ab}\gamma^{cd}\gamma^{ab}\chi, \\
&= \frac{1}{4}\partial_\mu\omega_{\nu ab}\gamma^{ab}\chi - \frac{1}{4}\partial_\nu\omega_{\mu ab}\gamma^{ab}\chi + \frac{1}{16}\omega_{\mu ab}\omega_{\nu cd}\gamma^{ab}\gamma^{cd}\chi - \frac{1}{16}\omega_{\mu ab}\omega_{\nu cd}\gamma^{cd}\gamma^{ab}\chi \\
&\stackrel{(c)}{=} \frac{1}{4}\partial_\mu\omega_{\nu ab}\gamma^{ab}\chi - \frac{1}{4}\partial_\nu\omega_{\mu ab}\gamma^{ab}\chi + \frac{1}{16}\omega_{\mu ab}\omega_{\nu cd}\gamma^{ab}\gamma^{cd}\chi - \frac{1}{16}\omega_{\mu ab}\omega_{\nu cd}\gamma^{cd}\gamma^{ab}\chi \\
&\stackrel{(d)}{=} \frac{1}{4}\partial_\mu\omega_{\nu ab}\gamma^{ab}\chi - \frac{1}{4}\partial_\nu\omega_{\mu ab}\gamma^{ab}\chi + \frac{1}{4}\omega_{\mu ba}\omega_{\nu cd}\eta^{ac}\gamma^{bd} - \frac{1}{4}\omega_{\nu ba}\omega_{\mu dc}\eta^{ad}\gamma^{bc}\chi \\
&\stackrel{(e)}{=} \frac{1}{4}R_{\mu\nu}{}^{mn}\gamma_{mn}\Psi_\rho^i.
\end{aligned}$$

The steps (a) and (b) use the definition (13).

The step (c) uses the relation  $\gamma^{ab}\gamma^{cd} - \gamma^{cd}\gamma^{ab} = -2\eta^{ac}\gamma^{bd} + 2\eta^{bc}\gamma^{ad} + 2\eta^{ad}\gamma^{bc} - 2\eta^{bd}\gamma^{ac}$ .

The step (d) uses the property  $\omega_{\mu ab} = -\omega_{\mu ba}$ .

The step (e) uses the definition (26).

Let's than calculate  $D_{[\mu}D_{\nu]}\gamma_\rho$ :

$$\begin{aligned}
D_{[\mu}D_{\nu]}\gamma_\rho &= D_{[\mu}D_{\nu]}(e_\rho^r\gamma_r), \\
&= D_{[\mu}D_{\nu]}e_\rho^r\gamma_r, \\
&= [D_\mu D_\nu e_\rho^r - D_\nu D_\mu e_\rho^r]\gamma_r, \\
&\stackrel{(a)}{=} [D_\mu(\partial_\nu e_\rho^r + \omega_\nu{}^r{}_k e_\rho^k) - D_\nu(\partial_\mu e_\rho^r + \omega_\mu{}^r{}_k e_\rho^k)]\gamma_r, \\
&\stackrel{(b)}{=} [\partial_\mu(\partial_\nu e_\rho^r + \omega_\nu{}^r{}_k e_\rho^k) + \omega_\mu{}^r{}_l(\partial_\nu e_\rho^l + \omega_\nu{}^r{}_l e_\rho^k) \\
&\quad - \partial_\nu(\partial_\mu e_\rho^r + \omega_\mu{}^r{}_k e_\rho^k) - \omega_\nu{}^r{}_l(\partial_\mu e_\rho^l + \omega_\mu{}^r{}_l e_\rho^k)]\gamma_r, \\
&= (\cancel{\partial_\mu\partial_\nu e_\rho^r}^a + \partial_\mu\omega_\nu{}^r{}_k e_\rho^k + \omega_\nu{}^r{}_k\cancel{\partial_\mu e_\rho^k}^b + \omega_\mu{}^r{}_l\cancel{\partial_\nu e_\rho^l}^c + \omega_\nu{}^r{}_l\omega_\mu{}^r{}_l e_\rho^k \\
&\quad - \cancel{\partial_\nu\partial_\mu e_\rho^r}^a - \partial_\nu\omega_\mu{}^r{}_k e_\rho^k - \omega_\mu{}^r{}_k\cancel{\partial_\nu e_\rho^k}^c - \omega_\nu{}^r{}_l\cancel{\partial_\mu e_\rho^l}^b - \omega_\nu{}^r{}_l\omega_\mu{}^r{}_l e_\rho^k)\gamma_r, \\
&\stackrel{(c)}{=} R_{\mu\nu}{}^{mn}\gamma_m e_{n\rho}.
\end{aligned}$$

The steps (a) and (b) use the definition (12).

The step (c) uses the definition (26).

## E.2 Calculation of $[\tilde{D}_\mu, \tilde{D}_\nu]\epsilon^i$

$$\begin{aligned}
[\tilde{D}_\mu, \tilde{D}_\nu]\epsilon^i &= \tilde{D}_\mu\tilde{D}_\nu\epsilon^i - \tilde{D}_\nu\tilde{D}_\mu\epsilon^i, \\
&\stackrel{(a)}{=} \tilde{D}_\mu(D_\nu\epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{ij}\gamma_\nu\epsilon^j) - \tilde{D}_\nu(D_\mu\epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{ij}\gamma_\mu\epsilon^j), \\
&\stackrel{(b)}{=} D_\mu(D_\nu\epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{ij}\gamma_\nu\epsilon^j) + \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{ik}\gamma_\mu(D_\nu\epsilon^k + \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{kj}\gamma_\nu\epsilon^j) \\
&\quad - D_\nu(D_\mu\epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{ij}\gamma_\mu\epsilon^j) - \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{ij}\gamma_\nu(D_\mu\epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}}\mathbf{M}_1^{ik}\gamma_\mu\epsilon^k)
\end{aligned}$$

$$\begin{aligned}
& -D_\nu(D_\mu\epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ij}\gamma_\mu\epsilon^j) - \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ik}\gamma_\nu(D_\mu\epsilon^k + \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{kj}\gamma_\mu\epsilon^j), \\
\stackrel{(c)}{=} & D_\mu D_\nu \epsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ij}D_\mu e_\nu^r \gamma_r \epsilon^j + \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ij}\gamma_\nu D_\mu \epsilon^j \xrightarrow{a} + \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ik}\gamma_\mu D_\nu \epsilon^k \xrightarrow{b} + \frac{\Lambda}{12}M_1^{ik}M_1^{kj}\gamma_\mu\gamma_\nu\epsilon^j \\
& -D_\nu D_\mu \epsilon^i - \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ij}D_\nu e_\mu^r \gamma_r \epsilon^j - \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ij}\gamma_\mu D_\nu \epsilon^j \xrightarrow{b} - \frac{\sqrt{\Lambda}}{2\sqrt{3}}M_1^{ik}\gamma_\nu D_\mu \epsilon^k \xrightarrow{a} - \frac{\Lambda}{12}M_1^{ik}M_1^{kj}\gamma_\nu\gamma_\mu\epsilon^j, \\
\stackrel{(d)}{=} & [D_\mu, D_\nu]\epsilon^i + \frac{\sqrt{\Lambda}}{4\sqrt{3}}M_1^{ij}D_{[\mu}e_{\nu]}^r \gamma_r \epsilon^j \pm \frac{\Lambda}{12}\gamma_\mu\gamma_\nu\epsilon^i \mp \frac{\Lambda}{12}\gamma_\nu\gamma_\mu\epsilon^i, \\
\stackrel{(e)}{=} & (\frac{1}{4}R_{\mu\nu}{}^{mn}\gamma_{mn} \pm \frac{\Lambda}{6}\gamma_{\mu\nu})\epsilon^i.
\end{aligned}$$

The steps (a) and (b) use the definition (51).

The step(c) takes into account  $D_\nu\gamma_\nu = D_\nu(e_\nu^r\gamma_r) = D_\nu e_\nu^r\gamma_r$ .

The step (d) takes into account  $(M_1)^2 = \pm I$  from (39) and  $D_{[\rho}e_{\sigma]}^s = 0$  obtained from (33) when  $\Psi_\mu^i = 0$ .

The step (e) takes into account the first result obtained in Appendix E.1 and the relation  $\gamma_{\mu\nu} = \frac{1}{2}\gamma_\mu\gamma_\nu - \frac{1}{2}\gamma_\nu\gamma_\mu$ .

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